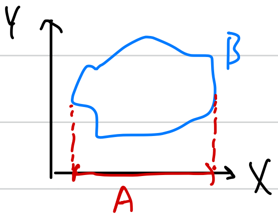
Measure Theory with Ergodic Horizons
Lecture 13

Universally measurable sets and functions.

Let X be a metric space. A set $B \subseteq X$ is called universally measurable if it is measurable with respect to every σ-finite Borel measure on X. Note that the collection $UM(X)$ of all universally measurable subsets of X is a σ-algebra (being the intersection of Meas_μ for all σ-finite Borel measures on X). For a metric space Y, a function $f : X \to Y$ is called universally measurable if it is $(UM(X), \mathcal{B}(Y))$-measurable, i.e. the f-preimage of each Borel set in Y is universally measurable.

Remark. Note that in the definition above, σ-finiteness can be replaced by finiteness, in other words, a set $B \subseteq X$ is universally measurable $\iff$ it is measurable with respect to every Borel probability measure. HW

It is a theorem in Descriptive Set Theory that all analytic sets are universally measurable, where a subset of a Polish space X is called analytic if it is the image of a Borel subset of a Polish space Y under a Borel function $f: Y \to X$. Equivalently, $A \subseteq X$ is analytic $\iff$ $\exists$ Polish Y and a Borel set $B \subseteq X \times Y$ s.t. $A = \mathrm{proj}_X(B)$.



The definition of universally measurable functions is motivated by the fact that measurable functions are not closed under composition (HW 6 Question 6):

Prop. Universally measurable functions are closed under composition: if X, Y, Z are metric spaces and $f: X \to Y$, $g: Y \to Z$ are universally measurable, then $g \circ f: X \to Z$ is universally measurable.

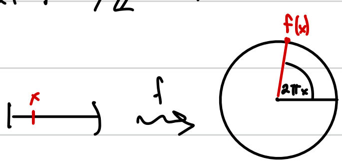
Proof. Uses push-forward measures and is left as HW.

Pushforward measures.

For measurable spaces $(X, \mathcal{I})$, $(Y, \mathcal{J})$, an $(\mathcal{I}, \mathcal{J})$-measurable function $f: X \to Y$, and a measure μ on $\mathcal{I}$, the pushforward of μ via f is the measure $f_* \mu$ on $\mathcal{J}$ defined by $f_* \mu(B) := \mu(f^{-1}(B))$ for each $B \in \mathcal{J}$. This measure is also sometimes denoted by μf^{-1}.

Examples.

(a) Let $f: \mathbb{R}/\mathbb{Z} \cong [0,1) \rightsquigarrow S^1 \subseteq \mathbb{C}$ be given by $x \mapsto e^{2\pi x i}$. Then the pushforward $f_* \lambda$ of Lebesgue measure λ via f is an invariant measure on S^1 under rotations, i.e. under the action of S^1 on itself by multiplication.



(b) Let $f: \mathbb{R} \rightsquigarrow \mathbb{R}_{>0} := (0, \infty)$ be the exponentiation function $x \mapsto e^x$. Then the pushforward $f_* \lambda$ of Lebesgue measure via f is invariant under scalar multiplication, i.e. the action of $(\mathbb{R}_{>0}, \cdot)$ on itself by translation.

In both of these examples, f is a group-isomorphism from $(\mathbb{R}/\mathbb{Z}, +)$ and $(\mathbb{R}, +)$ to $(S^1, \cdot)$ and $(\mathbb{R}_>, \cdot)$. Since the Lebesgue measure on $\mathbb{R}/\mathbb{Z} \cong [0,1)$ and on $\mathbb{R}$ is invariant under translation with respect to $+$, the pushforward measures via these isomorphisms are invariant under translation with respect to $\cdot$.

(c) Let $(X, \mathcal{I})$, $(Y, \mathcal{J})$ be measurable spaces and let μ be a measure on $(X \times Y, \mathcal{I} \otimes \mathcal{J})$. Then the pushforward measures $\mu_X := \mathrm{proj}_X{}_* \mu$ and $\mu_Y := \mathrm{proj}_Y{}_* \mu$ are called the marginals of μ and μ is called a joining of μ_X and μ_Y.

In probability, measurable functions to some metric space (sometimes $\mathbb{N}$) are called random variables and for a random variable $f:(\Omega,\mu)\to X$ the value $f(\omega)$ of some arbitrary ω is called a sample. The pushforward $f_*\mu$ via f is called the law of f.

Example. Let $\mathrm{Graph}(\mathbb{N}) :=$ the space of all graphs on $\mathbb{N}$, i.e. the vertex set is always $\mathbb{N}$ and the edge set is a subset of $\mathbb{N}^2$. This space can be identified with $2^{[\mathbb{N}]^2}$, where $[\mathbb{N}]^2$ is the set of all 2-element subsets of $\mathbb{N}$.
A random graph on $\mathbb{N}$ is a Borel probability measure on $\mathrm{Graph}(\mathbb{N})$. Most often a random graph is a pushforward measure $G_*\mu$ via some random variable $G:(\Omega,\mu)\to\mathrm{Graph}(\mathbb{N})$ and one may say "let G be a random graph on $\mathbb{N}$".
Given an operation $p:\mathrm{Graph}(\mathbb{N})\to\mathrm{Graph}(\mathbb{N})$, for example, cutting all cycles in a given graph, we would say that given a random graph G, we obtain another random graph $p(G)$. This just means that we switched from $G_*\mu$ to $(p\circ G)_*\mu$.

Invariant measures: Haar measure

Def. Let Γ be a group and $\Gamma\curvearrowright(X,\mathcal{B},\mu)$ be a measurable action of Γ on a measure space $(X,\mathcal{B},\mu)$, i.e. each $\gamma\in\Gamma$ is a $(\mathcal{B},\mathcal{B})$-measurable function. Then we say that this action preserves μ or that μ is Γ-invariant if $\gamma_*\mu=\mu$ for all $\gamma\in\Gamma$.
Note that this is equivalent to $\mu(\gamma\cdot B)=\mu(B)$ for each $\gamma\in\Gamma$.

In dynamics, given an action of a group Γ, one would like to have Γ-invariant probability measures to work with.

Important examples of invariant measures are Haar measures on locally compact groups.

Def. A topological group G is a group equipped with a topology which makes the group operation $\cdot : G \times G \to G$ and its inverse $()^{-1} : G \to G$ continuous.

Theorem (Haar). Every locally compact Hausdorff topological group G admits a unique (up to scaling) nonzero locally finite Borel measure ($\Leftrightarrow$ finite on compact sets) that is invariant under the left translation action of G on itself.